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Brief paper

Characterizing controllable subspace and herdability of signed weighted networks via graph partition ☆

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ABSTRACT

Herdability is a variant of controllability, and is an indicator of the ability to drive system states to a specific subset of the state space. This paper characterizes the controllable subspace and herdability of signed weighted networks. Specifically, a dynamic signed leader–follower network is considered, in which a small subset of the network nodes (i.e., the leaders) is endowed with exogenous control input and the remaining nodes are influenced by the leaders via the underlying network connectivity. The considered network permits positive and negative edges to capture cooperative and competitive interactions, resulting in a signed graph. Motivated by practical application, the system states are required to be driven by the leaders to be element-wise above a positive threshold, i.e., a specific subset rather than the entire state space as in classical controllability. Graph partitions are exploited to characterize the controllable subspace of the system, from which sufficient conditions are derived to render the system herdable. It is revealed that the quotient graph can be used to infer the herdability of the original graph, wherein criteria of the herdability of quotient graphs are developed based on positive systems. Examples are provided to illustrate the developed topological characterizations.

1. Introduction

Networked multi-agent systems are increasingly applied in biological science (Muldoon et al., 2016), social science (Kan, Klotz, Jr, & Dixon, 2015; Mirtabatabaei & Bullo, 2011), and engineering (Asimakopoulou, Dimeas, & Hatziargyriou, 2013; Klotz, Kan, Shea, Pasiliao, & Dixon, 2015; Klotz, Obuz, Kan, & Dixon, 2018). A subject of expanding interest, network controllability in these applications is indicative of the ability to arbitrarily control system states. When a networked system is fully controllable, the system states can be driven to any desired states. However, requiring a system to be fully controllable is often restrictive and unnecessary in many practical applications. For example, in applying adaptive cruise control to a platoon of autonomous vehicles, the vehicles are often required to maintain a desired positive speed. In a political election, a candidate aims to drive the supportive rate above a positive percentage in order to win. In these applications, fully controllable systems become unnecessary, since driving the vehicles' speed or the candidate's supportive rate to be negative does not make any physical sense. Instead, the relaxed controllability that drives the system states to a specific subset, rather than the entire state space, as in classical controllability, is of more practical significance. Such a relaxed controllability is referred to as herdability (Ruf, Egerstedt and Shamma, 2018). To this end, this work is practically motivated to characterize the herdability of networked systems from graph topological perspectives.

Since both herdability and controllability concern the ability to drive system states, the literature on the controllability of networked systems is first reviewed. Based on the type of interactions, a network can be classified as either cooperative or non-cooperative. Cooperative networks are often modeled as unsigned graphs in which only positive edge weights are allowed to represent cooperative interactions between network components, while non-cooperative networks are often modeled as signed graphs wherein both positive and negative edge weights represent cooperative and competitive interactions, respectively. The controllability of cooperative networks has long been a research focus. For instance, the influence of network topological structures on network controllability has been investigated using a variety of tools, such as graph theoretic approaches (Haghighi & Cheah, 2017; Liu, Slotine, & Barabási, 2011; Yazıcıoğlu, Abbas, & Egerstedt, 2016), structural controllability (Liu, Lin, & Chen, 2013a, 2013b; Tang, Wang, Gao, Qiao, & Kurths, 2014), and consensus based results (Aguilar & Gharesifard, 2015; Commault &

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Dion, 2013; Ji, Wang, Lin, & Wang, 2009). With respect to non-cooperative networks, the controllability of signed graphs has been investigated via structural balance in Alemzadeh, de Badyn, and Mesbahi (2017), Guan and Wang (2018), She, Mehta, Emily Doucette, and Kan (2019), She, Mehta, Ton, and Kan (2020) and Sun, Hu, and Xie (2017). Since network controllability is closely related to the underlying graph topology, graph partition (cf. Cardoso, Delorme, & Rama, 2007 and Godsil & Royle, 2001) was exploited to characterize network controllability from topological perspectives. In Egerstedt, Martini, Cao, Camlibel, and Bicchi (2012) and Rahmani, Ji, Mesbahi, and Egerstedt (2009), equitable partition and almost equitable partition were used to develop conditions that render cooperative networks (i.e., unsigned graphs) uncontrollable. In Aguilar and Gharesifard (2017), necessary conditions of uncontrollable networks were developed via almost equitable partition. These graph partition-based results were then extended to investigate non-cooperative networks (i.e., signed unweighted graphs) in Sun et al. (2017). Upper and lower bounds of the controllable subspace of networked systems were characterized based on graph partitions in OClery, Yuan, Stan, and Barahona (2013), Schaub et al. (2016) and Zhang, Cao, and Camlibel (2014). Despite substantial progress in characterizing network controllability, topological characterizations of herdability remain largely unknown.

Herdability, i.e., the ability to drive system states to a specific subset in the state space, was recently studied in the work of Ruf, Egerstedt et al. (2018). A positive subset of the state space was considered in Ruf, Egerstedt et al. (2018), where the system states are controlled to be element-wise above a positive threshold. Positive systems are a particular class of systems where, provided positive initial states, the states remain positive during evolution (Farina & Rinaldi, 2011). Necessary and sufficient conditions for a herdable positive networked system were developed based on controllable subspace and graph walks in Ruf, Egerstedt et al. (2018). The results of Ruf, Egerstedt et al. (2018) were then extended to characterize the herdability of linear systems based on sign patterns and graph structures in Ruf, Egersted and Shamma (2018) and Ruf, Egerstedt, and Shamma (2019).

Inspired by the works of Ruf, Egersted et al. (2018), Ruf, Egerstedt et al. (2018) and Ruf et al. (2019), this work considers the herdability of an undirected signed weighted networked system. Specifically, we consider a dynamic signed leader–follower network, where a small subset of the network nodes (i.e., the leaders) is endowed with exogenous control input and the remaining nodes are influenced by the leaders via the underlying network connectivity. The considered network allows positive and negative edges to capture cooperative and competitive interactions, resulting in a signed graph. Motivated by practical application, the system states are required to be driven by the leaders to be element-wise larger than a positive threshold, i.e., a specific subset rather than the entire state space as in classical controllability. To study the herdability of signed leader–follower networks, graph partitions are exploited to characterize the controllable subspace of the system, from which sufficient conditions are derived to render the system herdable. Examples are provided to illustrate the developed topological characterizations.

The contribution of this paper are multi-fold. First, this work characterizes the herdability of general undirected signed weighted networked systems. Topological characterizations of network herdability are developed using graph partitions. Since previous research mainly focused on unsigned graphs or signed unweighted graphs, generalized equitable partitions are developed in this work to take into account the edge weights of signed graphs. As an exception, signed weighted graph was partially investigated via graph partition in Aguilar and Gharesifard (2017). Differing from most graph partition-based results that consider Laplacian dynamics (Aguilar & Gharesifard, 2017; Egerstedt et al., 2012; OClery et al., 2013; Rahmani et al., 2009; Schaub et al., 2016; Sun et al., 2017; Zhang et al., 2014), this work considers the linear dynamics depending on the adjacency matrix of the underlying graph, which has many applications in nature (e.g. brain networks Gu et al., 2015) and man-made systems (e.g., multi-agent networks Jafari, Ajorlou, & Aghdam, 2011). The controllable subspace of such dynamics is then derived based on the generalized equitable partition. In addition, it is revealed that the quotient graph can be used to infer the herdability of the original graph, where herdability criteria of quotient graphs are developed based on positive systems. Since a quotient graph is an abstract representation of the original graph, the complexity of verifying the herdability of the original graph can be reduced simply by verifying the herdability of quotient graphs.

2. Problem formulation

Consider a networked system modeled by an undirected signed weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{v_1, \ldots, v_n\}$ denotes the node set and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ denotes the edge set. The interactions between nodes are captured by the weighted adjacency matrix $\mathcal{A} = \left[a_{ij}\right] \in \mathbb{R}^{n\times n}$, where $a_{ij} \neq 0$ if $\left(v_j, v_i\right) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. Differing from most existing research, self-loops are allowed in this work, i.e., $a_{ii} \neq 0$ for some $i \in \mathcal{V}$. The weight $a_{ij} \in \mathbb{R}$ takes real numbers, where $a_{ij} \in \mathbb{R}^+$ and $a_{ij} \in \mathbb{R}^-$ represent cooperative and competitive interactions between node v_i and v_j in the network, respectively. Let $\mathcal{A}_{:,i}$ and $\mathcal{A}_{i,:}$ denote the ith column and ith row of $\mathcal{A}$, respectively. The neighbor of node v_i is defined as $\mathcal{N}_i = \left\{ v_j \middle| \left(v_j, v_i\right) \in \mathcal{E}\right\}$ and the node degree of v_i is defined as $\beta_i = \sum_{j\in\mathcal{N}_i} a_{ij}$. We will show in subsequent sections that the designed node degree β_i facilitates characterization of the controllable space and herdability of signed graphs.

Let $x(t) = \left[x_1(t) \quad \ldots \quad x_n(t)\right]^T \in \mathbb{R}^n$ denote the stacked system states[1] of the network $\mathcal{G}$, where each entry $x_i(t) \in \mathbb{R}$ represents the state of node v_i. It is assumed that a subset $\mathcal{V}_l \subseteq \mathcal{V}$ of m nodes, referred to as leaders in the network, can be endowed with external controls. The remaining nodes $\mathcal{V}_f = \mathcal{V} \setminus \mathcal{V}_l$ are referred to as followers with $\mathcal{V}_l \cap \mathcal{V}_f = \emptyset$. Without loss of generality, the leaders' and the followers' indices are assumed to be $\mathcal{V}_l = \{1, \ldots, m\}$ and $\mathcal{V}_f = \{m+1, \ldots, n\}$. Suppose the system states evolve according to the linear dynamics

$$\dot{x}(t) = \mathcal{A}x(t) + Bu(t), \tag{1}$$

where $\mathcal{A} \in \mathbb{R}^{n\times n}$ is the adjacency matrix, $B = \left[e_1 \quad \ldots \quad e_m\right] \in \mathbb{R}^{n\times m}$ is the input matrix with basis vectors e_i, $i = 1, \ldots, m$, indicating that the ith node is endowed with external controls $u(t) \in \mathbb{R}^m$. Differing from most existing results that consider Laplacian dynamics, the dynamics (1) depends on the adjacency matrix $\mathcal{A}$, indicating a direct influence of the underlying network topology on the system dynamics.

The herdability of the system (1) is defined as follows.

Definition 1 (*Ruf et al., 2019*)**.** A networked system with dynamics in (1) is d-herdable if, for any $x(0) \in \mathbb{R}^n$, the system state $x(t)$ can be driven by a control input $u(t)$ to the set $H_d = \left\{x = \left[x_1 \quad \ldots \quad x_n\right]^T \in \mathbb{R}^n : x_i \geq d\right\}$ in finite time, where d is an arbitrary positive threshold.

Definition 1 implies a network is d-herdable if its states can be driven to a specific subset H_d of the state space. Throughout this work, the herdability of the system (1) is particularly referred

[1] Generalizations to multi-dimensional system states (e.g., $x_i \in \mathbb{R}^m$) are expected to be trivial via the matrix Kronecker product.

to the d-herdability. Recall that the controllability matrix $\mathcal{C} = \begin{bmatrix} B & \mathcal{A}B & \cdots & \mathcal{A}^{n-1}B \end{bmatrix}$ indicates the controllable subspace of a system (Hespanha, 2018). Therefore, the system (1) is completely controllable, if the controllability matrix $\mathcal{C} = \begin{bmatrix} B & \mathcal{A}B & \cdots & \mathcal{A}^{n-1}B \end{bmatrix}$ has full row rank. Since this work concerns driving $x(t)$ to H_d, the following lemma shows how the herdability of a system relates to the controllability matrix.

Lemma 1 (*Ruf, Egerstedt et al., 2018*)**.** *A networked system with dynamics in (1) is d-herdable if and only if there exists an element-wise positive vector $k \in \operatorname{Im}(\mathcal{C})$, where $\operatorname{Im}(\cdot)$ represents the range space of a matrix.*

As indicated in Lemma 1, the network herdability depends on the range space of the controllability matrix $\mathcal{C}$, which is closely related to the adjacency matrix $\mathcal{A}$ of the network. Motivated by this observation, the objective of this work is to develop characterizations of network herdability and its controllable subspace from graph topological perspectives. When not explicitly stated, the considered graphs are assumed to be undirected.

3. Characterizations of controllable subspace

This section presents topological characterizations of the controllable subspace of the system (1). Graph partition is used as the primary tool in exploring how network topology influences the herdability of signed networks. Particularly, Section 3.1 generalizes the classical definition of equitable partition of unsigned graphs to signed weighted graphs, from which Section 3.2 characterizes the controllable subspace of system (1). Section 3.3 discusses the construction of equitable partitions of leader–follower signed weighted graphs.

3.1. Generalized equitable partition

A graph can be partitioned into a set of cells, where each cell is a subset of the nodes. Let $\pi : \mathcal{V} \to \{C_1, C_2, \ldots, C_r\}$ denote a map that partitions the node set $\mathcal{V}$ into a set of distinct cells C_i, $i = 1, \ldots, r$, where $\cup_{i=1}^{r} C_i = \mathcal{V}$ and $C_i \cap C_j = \emptyset$ for $i \neq j$. Let $\beta_\pi\left(C_j, v_i\right) = \sum_{k \in \mathcal{N}_i \cap C_j} a_{ik}$ denote the cell-to-node degree of v_i from C_j under the partition π. Based on the partition π, we can construct a quotient graph $\mathcal{G}/\pi = (\mathcal{V}_\pi, \mathcal{E}_\pi, \mathcal{A}_\pi)$, where each cell C_i is treated as a node in $\mathcal{V}_\pi$ and $\left(C_j, C_i\right) \in \mathcal{E}_\pi$ represents a directed edge from C_j to C_i. Denote by $\mathcal{A}_\pi = \left[a_{ij}^{\pi}\right] \in \mathbb{R}^{r \times r}$ the adjacency matrix of the quotient graph $\mathcal{G}/\pi$ where the ijth entry of $\mathcal{A}_\pi$ representing the average edge weights from C_j to C_i is defined as

$$a_{ij}^{\pi} = \beta_\pi\left(C_j, C_i\right) = \frac{\sum_{v_i \in C_i} \beta_\pi\left(C_j, v_i\right)}{n_{C_i}}, \quad (2)$$

where n_{C_i} represents the number of nodes in C_i. Note that the $\mathcal{G}/\pi$ can be a directed graph, even if $\mathcal{G}$ is an undirected graph. Based on the defined cell-to-node degree, the equitable partition of signed weighted graphs is introduced.

Definition 2. Consider an undirected signed weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ and let $\pi = \{C_1, C_2, \ldots, C_r\}$ be a partition of $\mathcal{V}$. The partition π is a generalized equitable partition (GEP) if, for any two cells C_i and C_j where i and j are not necessarily distinct, it holds that $\beta_\pi\left(C_j, v_m\right) = \beta_\pi\left(C_j, C_i\right)$, $\forall v_m \in C_i$.

In the literature, the classical equitable partition is often defined based on the neighborhood of nodes, where an equitable partition indicates that, for any two cells C_i and C_j, all nodes in C_i have the same number of neighbors in C_j. Such a definition is applicable to unweighted unsigned graphs, since the edge weight a_{ij} only takes the value of 1 or 0, and, consequently, only the neighborhood of nodes matters in partitioning a graph. However, when considering signed weighted graphs, as in this work, the classical equitable partition is no longer applicable, since a_{ij} is a real number. Neighborhood of nodes alone are not sufficient to partition a signed weighted graph.

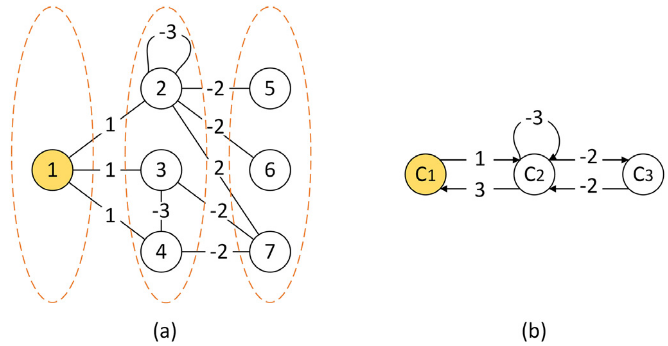

Fig. 1. (a) A non-trivial generalized equitable partition of $\mathcal{G}$. (b) The associated quotient graph $\mathcal{G}/\pi$.

In this work, Definition 2 generalizes the classical equitable partition by defining $\beta_\pi\left(C_j, v_i\right)$ which takes into account the real edge weights in graph partition. Definition 2 implies that, if π is a GEP, every node in C_i has the same cell-to-node degree from C_j. It is worth noting that the classical definition is a particular case of the designed GEP in Definition 2, since $\beta_\pi\left(C_j, v_i\right)$ reduces to indicate the number of neighbors of v_i in C_j if $a_{ij} \in \{0, 1\}$. In other words, any results developed based on GEP are immediately applicable to unweighted graphs. In addition, it should be noted that any graph $\mathcal{G}$ with n nodes admits a trivial GEP, i.e., $\pi = \{C_1, C_2, \ldots, C_n\}$, where each C_i is a singleton containing only v_i. Given a GEP π, the characteristic matrix of π is an $n \times r$ matrix $P(\pi) = \left[P_{ij}\right]$, where $P_{ij} = 1$ if $v_i \in C_j$ and $P_{ij} = 0$ if $v_i \notin C_j$. Clearly, the non-zero entries of the jth column of P indicate the node indices in the cell C_j.

Example 1. Fig. 1 illustrates the GEP π and the associated quotient graph $\mathcal{G}/\pi$. Fig. 1(a) shows a signed weighted graph $\mathcal{G}$ with 7 nodes and labeled edge weights. Node v_1 is the leader while the remaining nodes are the followers. A non-trivial generalized equitable partition is $\pi = \{C_1, C_2, C_3\}$, where $C_1 = \{v_1\}$, $C_2 = \{v_2, v_3, v_4\}$, $C_3 = \{v_5, v_6, v_7\}$. It can be verified that all nodes within the same cell have the same cell-to-node degree from another connected cell (including itself). Based on the partition π, the quotient graph $\mathcal{G}/\pi$ is shown in Fig. 1(b), where the numbers associated with the directed edges are the cell degree. Note that self-loop is allowed in Fig. 1(a) and (b). In addition, the graph $\mathcal{G}$ is undirected while the associated quotient graph $\mathcal{G}/\pi$ is directed. The characteristic matrix P of the GEP π and the adjacency matrix $\mathcal{A}_\pi$ of the quotient graph $\mathcal{G}/\pi$ are given by

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{A}_\pi = \begin{bmatrix} 0 & 3 & 0 \\ 1 & -3 & -2 \\ 0 & -2 & 0 \end{bmatrix}.$$

3.2. Controllable subspace

Based on the defined GEP, this section discusses how the graph partition characterizes the controllable subspace of the system (1). Considering a matrix E and a matrix T with appropriate dimensions, the vector space generated by the columns of E is called T-invariant if and only if there exists a matrix F such that $TE = EF$ (Godsil & Royle, 2001). That is, T maps any vector from $\operatorname{Im}(E)$ back to the same space $\operatorname{Im}(E)$.

Theorem 1. *Consider a signed weighted graph $\mathcal{G}$ with the adjacency matrix $\mathcal{A}$. Suppose $\pi = \{C_1, C_2, \ldots, C_r\}$ is a partition of $\mathcal{G}$. Let P and $\mathcal{A}_\pi$ denote the characteristic matrix of π and the adjacency matrix of the quotient graph $\mathcal{G}/\pi$. The partition π is a GEP of $\mathcal{G}$ if and only if $\mathcal{A}P = P\mathcal{A}_\pi$, i.e., $\operatorname{Im}(P)$ is $\mathcal{A}$-invariant.*

The proof of Theorem 1 is omitted here, since it is based on the well known results on classical equitable partitions of unweighted graphs and can be obtained by following a similar proof as in Cardoso et al. (2007). Previous research (cf. Aguilar & Gharesifard, 2017; Cardoso et al., 2007; Rahmani et al., 2009; Zhang et al., 2014) demonstrated that if a graph $\mathcal{G}$ has an almost equitable partition π, the range space of the associated characteristic matrix P is L-invariant, i.e., $LP = PL_\pi$, where L and L_π are the Laplacian matrices of the graph $\mathcal{G}$ and the quotient graph $\mathcal{G}/\pi$, respectively. Many results with respect to network controllability and controllable subspace were developed based on the condition $LP = PL_\pi$. However, it can be verified that $LP = PL_\pi$ does not in general hold for signed weighted graphs. For signed Laplacian dynamics defined in Altafini (2013), almost equitable partition does not guarantee L_π to be P-invariant. It should be noted that a particular Laplacian dynamics defined in Pan, Shao, and Mesbahi (2016) indeed admits $LP = PL_\pi$, under the condition that there exists an almost equitable partition.

To address this challenge, Theorem 1 reveals that the range space of P is $\mathcal{A}$-invariant if π is a GEP of $\mathcal{G}$, providing a means to characterize the controllable subspace of system (1) over signed graphs. It is well known that the controllable subspace of system (1) is

$$\operatorname{Im}(\mathcal{C}) = \operatorname{Im}(B) + \mathcal{A} \times \operatorname{Im}(B) + \cdots + \mathcal{A}^{n-1} \times \operatorname{Im}(B), \tag{3}$$

which is the smallest $\mathcal{A}$-invariant subspace that contains $\operatorname{Im}(B)$, where "+ " represents the union of two spaces (Hespanha, 2018). Since $\operatorname{Im}(P)$ is $\mathcal{A}$-invariant, Lemma 2 characterizes the controllable subspace from equitable partitions.

Definition 3. A GEP π is called leader-isolated generalized equitable partition (L-GEP) if every leader is a singleton cell in π.

Lemma 2. *Suppose π is an L-GEP of the signed weighted graph $\mathcal{G}$. The controllable subspace of system (1) satisfies $\operatorname{Im}(\mathcal{C}) \subseteq \operatorname{Im}(P)$, where $\mathcal{C}$ is the controllability matrix and P is the characteristic matrix of π.*

From the definition of the characteristic matrix P of an L-GEP and the input matrix B in (1), one has $\operatorname{Im}(B) \subseteq \operatorname{Im}(P)$. Lemma 2 then follows immediately, since

$$\begin{aligned}\operatorname{Im}(\mathcal{C}) &= \operatorname{Im}(B) + \mathcal{A} \times \operatorname{Im}(B) + \cdots + \mathcal{A}^{n-1} \times \operatorname{Im}(B) \\ &\subseteq \operatorname{Im}(P) + \mathcal{A} \times \operatorname{Im}(P) + \cdots + \mathcal{A}^{n-1} \times \operatorname{Im}(P) \\ &= \operatorname{Im}(P),\end{aligned}$$

where the fact that $\operatorname{Im}(P)$ is $\mathcal{A}$-invariant from Theorem 1 is used.

Lemma 2 provides an upper bound of the controllable subspace of system (1), which implies how the controllable subspace is related to the L-GEP of the graph. It should be noted that the L-GEP of $\mathcal{G}$ may not be unique. Let $\Pi = \{\pi_1, \pi_2, \ldots\}$ be the set of L-GEPs of $\mathcal{G}$, which implies that $\operatorname{Im}(B) \subseteq \operatorname{Im}(P)$ for every $\pi_i \in \Pi$. Consider two L-GEPs $\pi_1, \pi_2 \in \Pi$. The partition π_1 is said to be finer than π_2, denoted as $\pi_1 \leq \pi_2$, if each cell of π_1 is a subset of some cell of π_2. As demonstrated in Zhang et al. (2014),

$$\pi_1 \leq \pi_2 \Longleftrightarrow \operatorname{Im}(P_2) \subseteq \operatorname{Im}(P_1). \tag{4}$$

If $\pi_i \leq \pi^*$ for any $\pi_i \in \Pi$, then $\pi^* \in \Pi$ is referred to as the maximal L-GEP. In other words, π^* is the coarsest L-GEP, since the cells of any $\pi_i \in \Pi$ are subsets of the cells of π^*. Clearly, for any $\pi_i \in \Pi$,

$$\operatorname{Im}\left(P\left(\pi^*\right)\right) \subseteq \operatorname{Im}\left(P\left(\pi_i\right)\right) \tag{5}$$

obtains from (4). From (5), Lemma 3 is an immediate consequence.

Lemma 3. *The controllable subspace of system (1) can be upper bounded by $\operatorname{Im}(\mathcal{C}) \subseteq \operatorname{Im}(P(\pi^*))$, where π^* is the maximal L-GEP of $\mathcal{G}$.*

Compared with Lemma 2, a tighter upper bound of $\mathcal{C}$ derives from Lemma 3. For a system evolved with Laplacian dynamics, similar upper bounds of its controllable subspace were developed via almost equitable partition in Cardoso et al. (2007), Sun et al. (2017) and Zhang et al. (2014). As discussed previously, $\operatorname{Im}(P)$ is not L-invariant for signed graphs. Therefore, the upper bounds developed in Cardoso et al. (2007), Sun et al. (2017) and Zhang et al. (2014) are not applicable in this work.

3.3. Construction of π^*

The controllable subspace $\mathcal{C}$ is characterized based on the maximal L-GEP π^*. This section presents an algorithm to identify the maximal L-GEP π^*. The algorithm consists of the following steps:

Step 1: Let $\pi = \left\{C_1, \ldots, C_m, C_f\right\}$ be the initial partition, where $C_i = \{v_i\}$, $i = 1, \ldots, m$, indicates that each leader is a singleton cell in π and $C_f = \mathcal{V}_f$ represents the set of followers.

Step 2: For each node $v_i \in C_f$, calculate its node degree $\beta_i = \sum_{j\in\mathcal{N}_i} a_{ij}$, where a_{ij} is the weight associated with $\left(v_j, v_i\right) \in \mathcal{E}$. The nodes with the same node degree are then grouped into the same cell, i.e., $C_f = \left\{C_{m+1}, C_{m+2}, \ldots, C_q\right\}$. That is, the initial partition is refined by splitting C_f into a set of $q - m$ cells, where the nodes in each cell have the same node degree.

Step 3: For each node v_i in $C_j \in \pi$, $j \in \{m + 1, \ldots, q\}$, calculate the cell-to-node degree $\beta_\pi\left(C_p, v_i\right) = \sum_{k\in\mathcal{N}_i\cap C_p} a_{ik}$, $p \in \{1, \ldots, q\}$. Nodes with the same cell-to-node degree are grouped into the same cell and C_f is then updated based on the newly created cells such that the nodes in each cell have the same cell-to-node degree.

Step 4: Repeat Step 3 until no cells can be split.

Lemma 4. *Provided an undirected signed weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, the L-GEP π^* constructed from Steps 1–4 is maximal.*

Proof. By Definition 2, it is clear that the algorithm yields a generalized equitable partition since all nodes within the same cell have the same cell-to-node degree. The rest of the proof shows that the L-GEP is maximal. If the graph only contains trivial GEP, i.e., each cell only contains one node, the GEP obtained through the algorithm is indeed an L-GEP π^*. Suppose π_1 is a non-trivial L-GEP obtained from the algorithm and there exists another L-GEP π_2 such that π_2 contains π_1. That is, every cell in π_1 is a sub-cell of some cell in π_2. Suppose that two nodes v_p and v_q are in two different cells in π_1 but within the same cell in π_2. Based on Step 3, v_p and v_q are separated in two different cells if and only if there exists a cell C_j, such that $\beta_\pi\left(C_j, v_p\right) \neq \beta_\pi\left(C_j, v_q\right)$. However, this condition contradicts the fact that v_p and v_q are in the same cell in π_2. Hence, π_1 constructed from the algorithm is guaranteed to be maximum.

4. Characterizations of network herdability

Using the characterized controllable subspace, the herdability of system (1) is investigated in this section. Section 4.1 reveals that the herdability of $\mathcal{G}$ can be verified based on its quotient graph $\mathcal{G}/\pi$. Section 4.2 presents the methods for verifying network herdability from positive systems' perspectives.

4.1. Herdability via quotient graphs

Consider a quotient graph $\mathcal{G}/\pi$ constructed from an L-GEP $\pi = \{C_1, \ldots, C_r\}$, where the first m cells are the leader cells with the follower cells indexed from $m+1$ to r. When considering Laplacian dynamics (cf. Egerstedt et al., 2012; OClery et al., 2013; Schaub et al., 2016), due to the consensus properties of graph Laplacian, the average of system states in each cell, i.e., $\bar{x}_i = \sum_{j\in C_i} x_j$, is often used to yield a compact system representation $\dot{\bar{x}}(t) = L_\pi \bar{x}(t) + B_\pi u_\pi(t)$. Since such a dynamics is no longer valid to represent the evolution of system states when considering adjacency dynamics (1), the sum of states in each cell is used instead. Let $\hat{x} = \begin{bmatrix}\hat{x}_1 & \ldots & \hat{x}_r\end{bmatrix}^T \in \mathbb{R}^r$ denote the stacked states, where each entry $\hat{x}_i = \sum_{j\in C_i} x_j$ represents the sum of the states in C_i. The system (1) can then be rewritten in a compact form of

$$\dot{\hat{x}}(t) = \mathcal{A}_\pi \hat{x}(t) + B_\pi u_\pi(t), \tag{6}$$

where $u_\pi \in \mathbb{R}^m$ is the control input reorganized from u in (1) based on the L-GEP π, and $B_\pi = \begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix} \in \mathbb{R}^{r\times m}$ is the input matrix with basis vectors e_i indicating the ith cell is the leader cell endowed with external controls u_π. To see that, consider the ith row of (6),

$$\begin{aligned}\dot{\hat{x}}_i(t) &= \sum_{j\in\{1,\ldots,r\}} a^\pi_{ij}\hat{x}_j(t) + B^\pi_{i,:}u_\pi(t)\\ &= \sum_{j\in\{1,\ldots,r\}} \beta_\pi\left(C_j, C_i\right)\hat{x}_j(t) + B^\pi_{i,:}u_\pi(t),\end{aligned}$$

where $B^\pi_{i,:}$ represents the ith row of B_π. From (1), summing the dynamics of the nodes $v_i \in C_i$, one has

$$\begin{aligned}\sum_{v_i\in C_i}\dot{x}_i &= \sum_{v_i\in C_i}\left(\sum_{v_j\in\mathcal{V}} a_{ij}x_j + B_{i,:}u(t)\right)\\ &= \sum_{v_i\in C_i}\left(\sum_{v_j\in C_1} a_{ij}x_j + \cdots + \sum_{v_j\in C_r} a_{ij}x_j \right.\\ &\quad \left. + B_{i,:}u(t)\right),\end{aligned}$$

where $\sum_{v_i\in C_i}\sum_{v_j\in C_j} a_{ij}x_j = \beta_\pi\left(C_j, C_i\right)\hat{x}_j(t)$ and $\sum_{v_i\in C_i} B_{i,:}u(t) = B^\pi_{i,:}u_\pi(t)$. Therefore, (6) is an equivalent representation of the system (1).

Theorem 2. *Consider a system (1) evolving over a signed weighted graph $\mathcal{G}$. Let π be an L-GEP of $\mathcal{G}$, which yields a quotient graph $\mathcal{G}/\pi$ with dynamics (6). The system (1) over $\mathcal{G}$ is d-herdable if the system (6) over $\mathcal{G}/\pi$ is d-herdable.*

Proof. If the system (6) over the quotient graph $\mathcal{G}/\pi$ is d-herdable, $\hat{x}$ can be driven by u_π element-wise above an arbitrary positive threshold d. Since each entry $\hat{x}_i$ can be driven above d, there must exist at least one node $v \in C_i$ whose state is positive. Let $\mathcal{K}$ denote the set of state indices for which $x_i > 0$, $\forall i \in \mathcal{K}$. Thus, we can construct a vector $k \in \operatorname{Im}(\mathcal{C})$ such that $k_i > 0$, $\forall i \in \mathcal{K}$, since v_i is d-herdable (Ruf, Egerstedt et al., 2018). Note that, for each cell in π, there exists at least one node whose corresponding entry in k is positive. According to Lemma 2, $k \in \operatorname{Im}(\mathcal{C})$, $\operatorname{Im}(\mathcal{C}) \subseteq \operatorname{Im}(P)$, leading to $k \in \operatorname{Im}(P)$. Since the columns in P are linearly independent and the nonzero entries are all ones, $k \in \operatorname{Im}(P)$ indicates that the entries in k corresponding to the nodes in the same cell must all be positive. Therefore, $k \in \operatorname{Im}(\mathcal{C})$ is an element-wise positive vector. By Lemma 1, the system (1) evolving over $\mathcal{G}$ is d-herdable. □

Verifying network herdability based on Lemma 1 can be challenging, since it needs to check the existence of an element-wise positive vector $k \in \operatorname{Im}(\mathcal{C})$. Such a method can be significantly challenging, or even prohibitive, when addressing large-scale networks. Theorem 2 suggests that, rather than directly verifying the herdability of $\mathcal{G}$, the quotient graph $\mathcal{G}/\pi$ can be exploited. The graph $\mathcal{G}/\pi$ can be viewed as an abstract representation of $\mathcal{G}$ that captures the key topological structure of $\mathcal{G}$ while preserving its certain properties (e.g., herdability). Since $\mathcal{G}/\pi$ is more compact than $\mathcal{G}$ in terms of system dimensions, Theorem 2 provides an efficient means to investigate the herdability of $\mathcal{G}$.

4.2. Herdability of positive systems

Since $\mathcal{G}/\pi$ can be used to verify network herdability, this section presents verification methods based on positive systems.

Definition 4 (*Farina & Rinaldi, 2011*)**.** A dynamical system $\dot{x}(t) = f(x(t), t)$, $x \in \mathbb{R}^n$, is a positive system if $x(0) \geq 0_n$ implies $x(t) \geq 0_n$ for all t.

Definition 4 indicates positive systems are a class of systems in which the states remain non-negative during evolution, provided the initial states are non-negative. As indicated in De Leenheer and Aeyels (2001), an affine system $\dot{x} = Ax + b$ is a positive system if the system matrix $A \in \mathbb{R}^{n\times n}$ is a Metzler matrix[2] and $b \in \mathbb{R}^n$ is element-wise positive. Therefore, if system (1) evolves on an unsigned graph (i.e., the associated adjacency matrix $\mathcal{A}$ is a Metzler matrix), system (1) is positive, since there always exist control inputs u for leaders such that Bu is element-wise positive. Ensuing from Ruf, Egerstedt et al. (2018), Lemma 5 characterizes how positive system infers network herdability:

Lemma 5 (*Ruf, Egerstedt et al., 2018*)**.** *A positive system is completely d-herdable if and only if it is input connectable, i.e., there exist (directed) paths from leaders to followers.*

However, when evolving on signed graphs, the system matrix $\mathcal{A}$ in (1) is no longer a Metzler matrix due to potential negative edge weights. Theorem 3 represents how the quotient graph can be exploited to verify network herdability.

Theorem 3. *Consider a system (1) evolving over a signed weighted graph $\mathcal{G}$. Let π be an L-GEP of $\mathcal{G}$, which yields a quotient graph $\mathcal{G}/\pi$ with dynamics (6). The system (1) over $\mathcal{G}$ is d-herdable if $\mathcal{G}/\pi$ is input connected (i.e., follower cells are reachable from leader cells) and the cell degree between distinct cells in $\mathcal{G}/\pi$ is non-negative.*

Proof. Consider the quotient graph $\mathcal{G}/\pi$. If the cell degree between distinct cells in $\mathcal{G}/\pi$ is non-negative, the matrix $\mathcal{A}_\pi$ is a Metzler matrix. Note that $\mathcal{A}_\pi$ remains a Metzler matrix if the cell degree with respect to itself is negative (i.e., self-loop with negative weight). Per Definition 4, system (6) is a positive system since there always exist control inputs u_π for leader cells such that $B_\pi u_\pi$ is element-wise positive. In addition, if $\mathcal{G}/\pi$ is input connected, system (6) is d-herdable by Lemma 5. Therefore, system (1) on $\mathcal{G}$ is also d-herdable by Theorem 2. □

[2] A matrix is called a Metzler matrix if all its off-diagonal entries are non-negative.

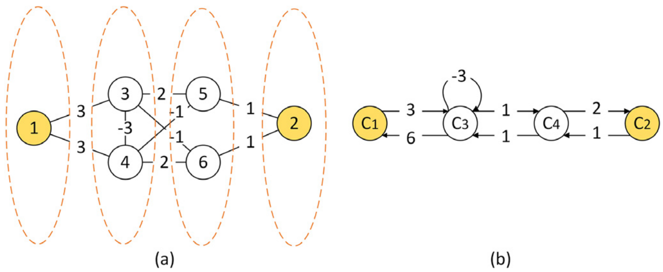

Fig. 2. (a) A non-trivial generalized equitable partition of $\mathcal{G}$, where $C_1 = \{v_1\}$ and $C_2 = \{v_2\}$ are leader cells, and $C_3 = \{v_3, v_4\}$ and $C_4 = \{v_5, v_6\}$. (b) The system (6) evolving on $\mathcal{G}/\pi$ is a positive system.

Theorem 3 indicates that, even if $\mathcal{G}$ is signed with negative edge weights, $\mathcal{G}$ is d-herdable, provided $\mathcal{G}/\pi$ satisfies the conditions stated in Theorem 3. In other words, in order to verify the herdability of $\mathcal{G}$, we only need to determine if $\mathcal{G}/\pi$ is input connectable and the cell degree between distinct cells in $\mathcal{G}/\pi$ is non-negative.

Example 2. To illustrate Theorem 3, consider a signed graph $\mathcal{G}$ as shown in Fig. 2(a). The nodes v_1 and v_2 are assumed to be leaders while the remaining nodes as followers. Let $\pi = \{C_1, C_2, C_3, C_4\}$ be an L-GEP of $\mathcal{G}$, where $C_1 = \{v_1\}$, $C_2 = \{v_2\}$, $C_3 = \{v_3, v_4\}$, $C_4 = \{v_5, v_6\}$. The edge weights are labeled along the edges. Due to the existence of negative edge weights, the adjacency matrix $\mathcal{A}$ is not a Metzler matrix; consequently, it cannot be determined if system (1) is a positive system. To verify the herdability of $\mathcal{G}$, the quotient graph $\mathcal{G}/\pi$ is constructed as shown in Fig. 2(b). Since $\mathcal{A}_\pi$ is a Metzler matrix and the leader cell has directed paths to all other cells (i.e., input connectable), system (6) is d-herdable, and therefore $\mathcal{G}$ is d-herdable.

5. Conclusion

The herdability of signed weighted graphs is investigated in this work, wherein graph partitions are exploited to characterize topological structures to ascertain network herdability. Generalized equitable partitions are developed to take into account the edge weights of signed graphs. The controllable subspace of such dynamics is then derived based on the generalized equitable partition. It is discovered that the quotient graph can be used to infer the herdability of the original graph, for which criteria for herdability of quotient graphs are developed based on positive systems. Additional research will consider graph partition on signed directed graphs or other graph-theoretic methods to characterize the herdability of general signed graphs.

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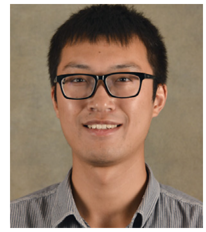

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